Int. J. Solids Structures Vol. 30, No. 11, pp. 1527–1532, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00

A NOTE ON CURVED CRACKED BEAMS

W. H. MÜLLER
Failure Analysis Associates, Inc., Engineering and Scientific Services,
149 Commonwealth Drive, P.O. Box 3015, Menlo Park, CA 94025, U.S.A.

and

G. HERRMANN and H. GAO
Division of Applied Mechanics, Durand Building, Stanford University,
Stanford, CA 94305-4040, U.S.A.

(*Received* 16 *September* 1991; *in revised form* 16 *December* 1992)

Abstract—This paper is concerned with evaluating stress intensity factors for a curved cracked beam of small to moderate curvature using a simple engineering method which allows us to estimate the strain energy release rate based on elementary beam theory. The basic methodology is briefly discussed and an example application to a cracked circular beam gives results which are in reasonable agreement with more accurate numerical calculations in the literature.

1. INTRODUCTION

Curved beams have many applications in structures like roofs, pipes, bridges and reactor vessels. Figure 1 illustrates a cracked circular beam which is often used in fracture testing of shell-like structural elements. In order to obtain reliable K_{IC} data it is necessary to calculate the stress intensity factor (SIF) values within a broad range of crack length and specimen dimension. So far only a numerical method, the so-called mapping-collocation technique [e.g. Andrasic and Parker (1980), Tracy (1975) and Kapp *et al.* (1980)], has been used to determine solutions for commonly-used geometries and loading conditions, e.g. moments and/or tensile forces applied to the ends of a circular cracked beam. The SIF database obtained in this manner has been limited to a certain range of geometrical and loading parameters and it is by no means straightforward how to extend these bounds without performing additional numerical calculations.

Simple engineering methods which allow a fast but approximate determination of critical design variables such as the stress intensity factor are invaluable to a design engineer who "must strike a balance between time, cost and accuracy in selecting a method" (Andrasic and Parker, 1980, p. 67). With these thoughts, Herrmann and coworkers (Herrmann and Sosa, 1986; Kienzler and Herrmann, 1986a, b) developed an engineering approach of using elementary beam theory to estimate the strain energy release rate due to enlargement of a crack and to calculate the stress intensity factor. Gao and Herrmann

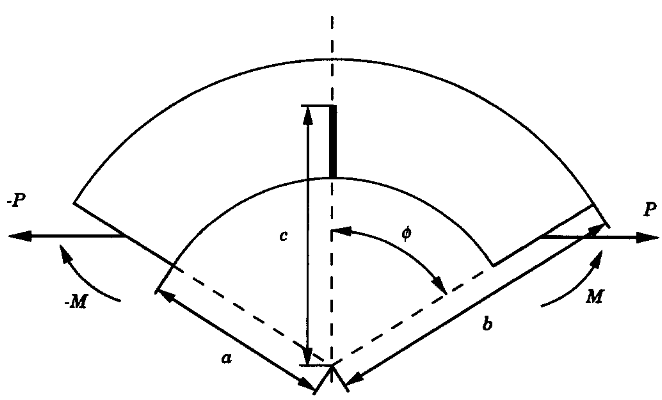

Fig. 1. Circular cracked beam subjected to a moment couple or tensile forces.

(1992) showed that the SIF predicted from this method can be further made to match asymptotically with standard crack solutions in limiting cases, and, in that procedure, a correction factor, β, can be determined which leads to more accurate SIFs over the full range of crack length.

This paper has the intention of exploring the applicability of the beam-theory method in problems involving curved cracked beams. For conciseness we shall focus our discussion on a single-edge-notched circular beam of rectangular cross-section subjected to a bending moment or a tensile force. It is found that the simple beam-theory method is indeed capable of providing results which compare favorably with numerical data available in the literature. It is relatively easy to extend the basic methodology discussed here to more complicated cracked beam problems.

2. THE METHOD

The energy release rates $\partial U/\partial l$ and $\partial U/\partial h$ of cracks which are either extended by $\mathrm{d}l$ or widened by $\mathrm{d}h$ are related as follows (Gao and Herrmann, 1992):

$$\frac{\partial U}{\partial l} = 2\beta \frac{\partial U}{\partial h}. \tag{1}$$

The strain energy of the body is denoted by U and β is a correction factor depending upon the crack size and the geometry of the specimen.

In this paper we deal exclusively with the state of plane stress, in which case the energy release rate is related to the mode I stress intensity factor, K_{I}, as follows:

$$\frac{K_{\mathrm{I}}^2}{E} = -\frac{1}{t}\frac{\partial U}{\partial l} \Rightarrow K_{\mathrm{I}} = \sqrt{-\frac{2\beta E}{t}\frac{\partial U}{\partial h}}. \tag{2}$$

Here, E is Young's modulus and t stands for the thickness of the beam.

Obviously, two unknowns must be determined in order to arrive at an explicit expression for the SIF of a cracked beam: $\partial U/\partial h$ and β. In order to determine the first unknown, U, consider the beams with discontinuities in stiffness as indicated in Fig. 1. Following Herrmann and coworkers (Herrmann and Sosa, 1986; Kienzler and Herrmann, 1986a, b), $\partial U/\partial h$ can be obtained by calculating the difference between the strain energy stored in the actual cracked cross-section and that in the uncracked counterpart. When this is done we can show that

$$\frac{\partial U}{\partial h} = -\frac{t}{c}\left[\int_c^b u_c(r,\phi) r\,\mathrm{d}r - \int_a^b u(r,\phi) r\,\mathrm{d}r\right], \tag{3}$$

where $u_c(r, \phi)$, $u(r, \phi)$ denote the strain energy density at the cracked and uncracked cross-section, respectively (Fig. 1).

The factor β in eqn (2) can be determined provided asymptotic expansions for the SIFs of short and long cracks are known (Gao and Herrmann, 1992). The expansions depend upon the loading conditions and upon the geometry of the beam in question. The asymptotic solutions for straight beams were presented by Benthem and Koiter (1973), but, unfortunately, no solutions are seemingly available for curved beams. However, it is found that the curvature of the beam (which should remain sufficiently small within the framework of elementary beam theory) plays a negligible role in the correction factor and is sufficiently represented by the energy release rate expression in eqn (3). Thus in our calculations we shall use the known expressions for β of straight beams.

3. SIFs OF CIRCULAR BEAMS OF RECTANGULAR CROSS-SECTION

3.1. *Pure bending*

Consider the cracked, circular beam shown in Fig. 1 subjected to a bending moment, M, at its ends. The solution for the stresses in such a beam, given in Timoshenko and Goodier (1970), can be used to perform the integrations in (3) leading to:

$$K_\mathrm{I} = \frac{4M\sqrt{\beta_\mathrm{M}}}{tb^{3/2}}\sqrt{\frac{1}{2x}\left\{\frac{1-x^2}{(1-x^2)^2-(2x\ln x)^2}-\frac{1-y^2}{(1-y^2)^2-(2y\ln y)^2}\right\}},$$

$$x = \frac{c}{b},\quad y = \frac{a}{b}. \tag{4}$$

Figure 2 presents numerical values for the stress intensity factor, K_I, as predicted by eqn (4) and normalized by K_0 [cf. Andrasic and Parker (1980) or Tracy (1975)]:

$$K_0 = \frac{4M\sqrt{\pi}}{tb^{3/2}}\frac{(-2\ln y-1+y^2)\sqrt{x-y}}{(1-y^2)^2-(2y\ln y)^2}. \tag{5}$$

The correction factor β_M has been chosen to be the same as that for the straight beam under pure bending (Gao and Herrmann, 1992), i.e.

$$\beta_\mathrm{M} = 1.3183\left\{1+2\left(\frac{1-x}{1-y}\right)^{6.65}\right\}. \tag{6}$$

As shown in Fig. 2, the results from (4)–(6) agree well with numerical data obtained by mapping collocation (Andrasic and Parker, 1980; Tracy, 1975). Usually the difference is much less than 5%, only in the case of $y = 1/1.25$ can it go as high as 16%.

Next, consider the curved three-point bending specimen shown in Fig. 3, in which case the net moment at the cracked cross-section is given by:

$$M = \frac{P}{2}a\sin\phi. \tag{7}$$

Figure 4 shows the SIFs resulting from eqns (4), (6) and (7) for the case $\phi = \pi/4$. The results have been normalized by:

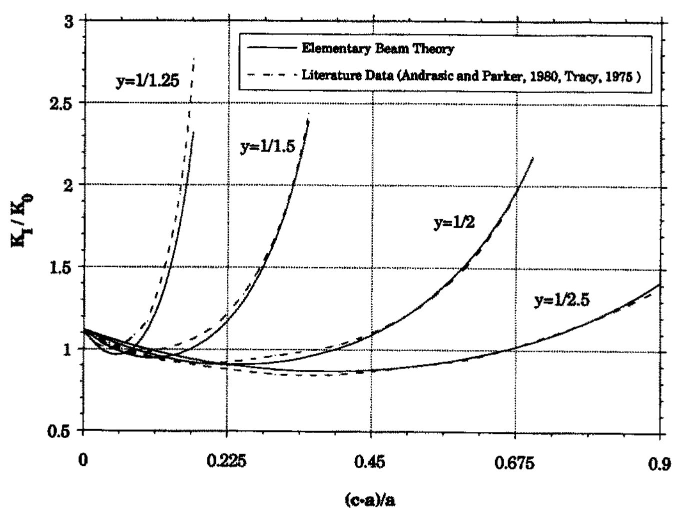

Fig. 2. Correction functions for a curved cracked beam (pure bending).

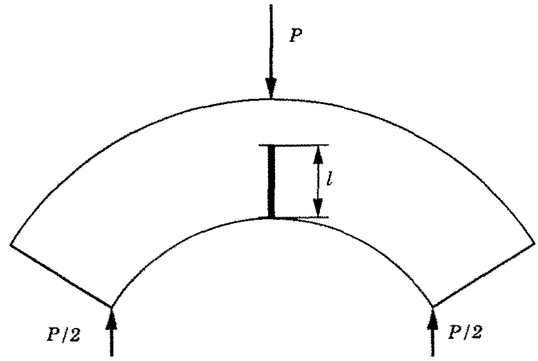

Fig. 3. Curved three-point bending specimen.

$$K_0 = \frac{P}{t} \frac{[\pi(x-y)]^{1/2}}{\sqrt{b}(1-y)}. \tag{8}$$

As in the case of pure bending the correction function, β_M, for the straight beam leads to SIFs which agree satisfactorily with the data presented in the literature (Tracy, 1975). Similar to the curve for $y = 1/1.25$ in Fig. 2 the difference reaches a maximum of 11%.

In the limiting case of a straight beam one may verify that eqn (4) reduces to the corresponding equation given by Herrmann and Sosa (1986).

3.2. *Tensile loading*

Consider the cracked, circular beam of Fig. 1 which is now subjected to a pair of tensile forces of strength, P, applied at the center of its end cross-section.

The stress intensity factor, K_I, consists of two parts, one due to the resulting moment and one due to the influence of the tensile forces. We shall denote these parts by K_M and K_P, respectively:

$$K_I = K_M + K_P. \tag{9}$$

To compute both parts two simplifications will be made. First, we shall assume that the neutral axis is located in the middle of the ligament, independently of the crack depth. As was pointed out by Benthem and Koiter (1973) this is not true for cracks which run almost through the beam. However, in comparison with numerical results we shall see that this

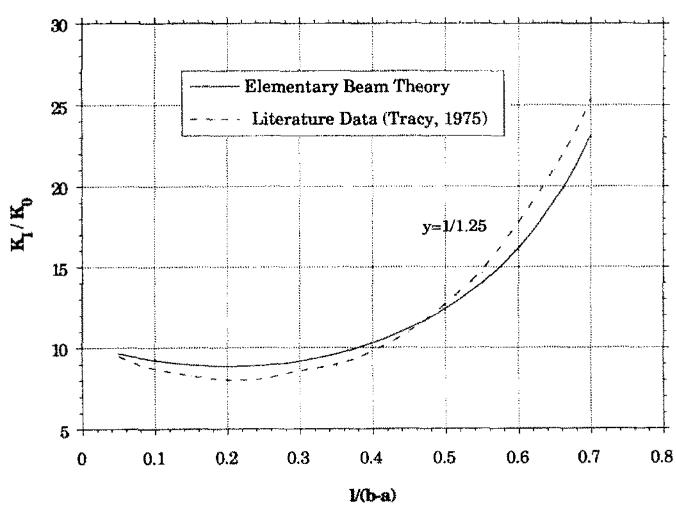

Fig. 4. Correction functions for a curved cracked beam (three-point bending).

effect is negligible. Thus we shall use eqns (4) and (6) together with the following expression for the moment:

$$M = \frac{Pb}{2}\{1+x-(1+y)\cos\phi\}. \tag{10}$$

Second, we note that for curved beams the moment contributes mostly to the stress intensity factor and that it suffices to use the corresponding equations of K_P for straight beams which, according to Kienzler and Herrmann (1986a, b), Herrmann and Gao (1992), and Müller *et al.* (1993), read:

$$K_P = \frac{P}{t\sqrt{b}}\frac{\sqrt{\beta_P}}{\sqrt{1-y}}\sqrt{\frac{x-y}{1-x}},$$

$$\beta_P = 3.955\left(\frac{1-x}{1-y}\right)^3 + 26.797\left(\frac{x-y}{1-y}\right)^2. \tag{11}$$

Figure 5 presents a numerical evaluation of eqn (9) for $\phi = \pi/4$ and $\pi/2$ when normalized by K_0:

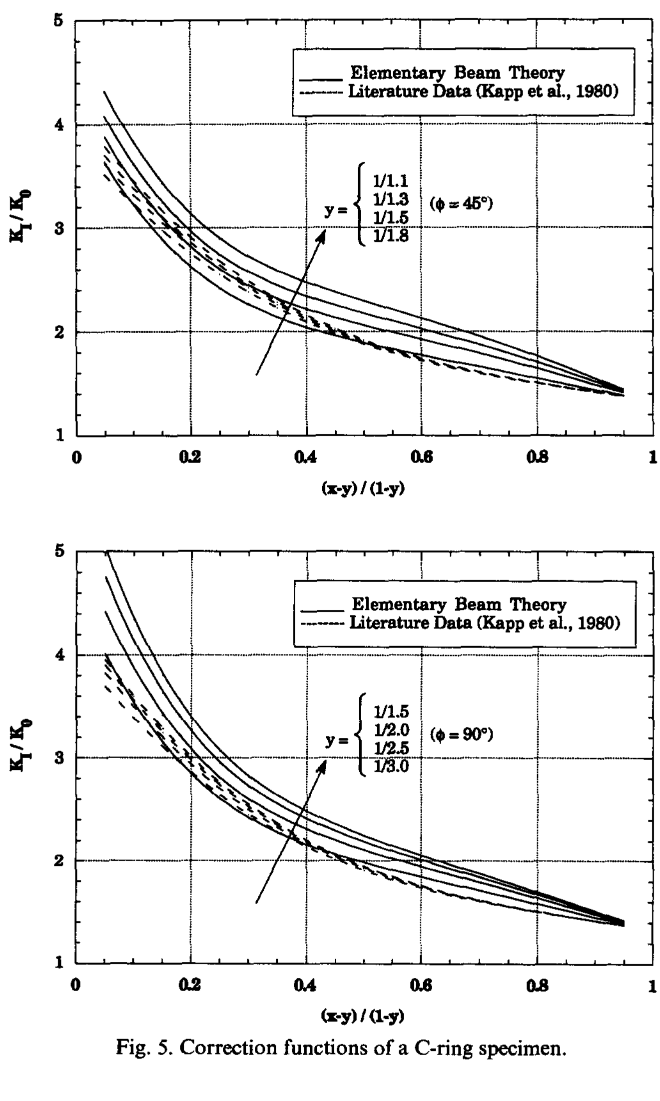

Fig. 5. Correction functions of a C-ring specimen.

$$K_0 = \frac{P}{t\sqrt{b}} \frac{\sqrt{x-y}}{\sqrt{1-y}(1-x)^{3/2}} [1.9+1.1x-1.5(1+y)\cos\phi]. \tag{12}$$

It can be compared to the following formula, based on curve fitting of mapping-collocation data (Kapp *et al.*, 1980):

$$K_1 = K_0 f\left(\frac{x-y}{1-y}\right) g\left(y, \frac{x-y}{1-y}\right), \tag{13}$$

where

$$f\left(\frac{x-y}{1-y}\right) = 3.74-6.3\frac{x-y}{1-y}+6.32\left(\frac{x-y}{1-y}\right)^2-2.43\left(\frac{x-y}{1-y}\right)^3,$$

$$g\left(y, \frac{x-y}{1-y}\right) = 1+0.25\frac{(1-x)^2}{1-y}. \tag{14}$$

Use of formula (13) is accurate within 3% when limited to the following range of parameters:

$$0.2 \leqslant \frac{x-y}{1-y} \leqslant 1.0, \quad 0 \leqslant \frac{y-0.5(1-y)\cos\phi}{1-y} \leqslant 1.0. \tag{15}$$

However, in order to obtain Fig. 5, eqns (13), (14) were evaluated regardless of this limit. Note that even then the resulting data agree fairly well with those predicted by elementary beam theory.

4. OUTLOOK AND CONCLUSIONS

Elementary beam theory was applied to derive analytical expressions for the SIFs of cracked circular beams which are used as structural elements and for fracture toughness testing of shell structures. The results obtained show an agreement with existing solutions. Wherever the range of application of a particular former result in the literature was limited to a certain range of crack length or load positioning, an extended solution has been provided. Finally, it should be mentioned that the presented method can be used to investigate a great variety of fracture mechanics problems, such as the determination of SIFs in specimens of orthotropic and composite materials or cracked pipes and shells. Papers concerned with these topics are in preparation.

Acknowledgement—The principal part of this work was completed while one of the authors (W.H.M.) was a Max-Kade fellow at Stanford University and at the University of California at Santa Barbara. Partial support by a DOE- and an AFOSR-grant is also gratefully acknowledged.

REFERENCES

Andrasic, C. P. and Parker, A. P. (1980). Weight functions for cracked, curved beams. In *Numerical Methods in Fracture Mechanics* (Edited by D. R. F. Owen and A. R. Luxmoore), pp. 67–82. Swansea.

Benthem, J. P. and Koiter, W. T. (1973). Asymptotic approximations to crack problems. In *Methods of Analysis and Solutions of Crack Problems* (Edited by G. C. Sih), pp. 131–177. Noordhoff, Leyden.

Gao, H. and Herrmann, G. (1992). On estimates of stress intensity factors for cracked beams and pipes. *Engng Fract. Mech.* **41**(5), 695–706.

Herrmann, S. and Sosa, H. (1986). On bars with cracks. *Engng Fract. Mech.* **24,** 889–894.

Kapp, J. A., Newman, Jr, J. C. and Underwood, J. H. (1980). A wide range stress intensity factor expression for the C-shaped specimen. *J. Test Eval., JTEVA* **8**(6), 314–317.

Kienzler, R. and Herrmann, G. (1986a). An elementary beam theory of defective beams. *Acta Mech.* **62,** 37–46.

Kienzler, R. and Herrmann, G. (1986b). On material forces in elementary beam theory. *J. Appl. Mech.* **108,** 561–564.

Müller, W. H., Herrmann, G. and Gao, H. (1993). Elementary strength theory of cracked beams. *J. Theoret. Appl. Fract. Mech.* (in press).

Timoshenko, S. P. and Goodier, J. N. (1970). *Theory of Elasticity* (3rd Edn). McGraw-Hill, Maidenhead.

Tracy, P. G. (1975). Analysis of a radial crack in a circular ring segment. *Engng Fract. Mech.* 7, 253–260.